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ASYMPTOTIC SOLUTION OF THE PROBLEM OF THE OPTIMAL CONTROL OF NON-LINEAR OSCILLATIONS IN THE NEIGHBOURHOOD OF A RESONANCE†

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A weakly controllable system with two rotating phases is considered in a regime of resonance oscillations. The characteristic rate of change of the slow variables in the system is of the order of ε, and the control is contained in the terms of the equations of order $\varepsilon^{3/2}$. This order of magnitude of the control makes it possible to control a resonance regime over time intervals of the order of $1/\varepsilon$. The purpose of the control is to minimize a functional representing the deviation from a steady resonance regime. It is shown that there is a hierarchy of fast and slow motions in the equations of the maximum principle. An algorithm is described for the asymptotic integration of these equations using successive averaging. The problem of vibrational maintenance of the steady rotation of an unbalanced rotor is considered as an example. © 1999 Elsevier Science Ltd. All rights reserved.

In previous investigations of controllable non-linear systems [1–3] it was assumed that the system is functioning outside the resonance region or passes through resonances without "capture". The problem of controlling the motion in the neighbourhood of a resonance in a quasi-linear system with slowly varying frequency has also been discussed [4]. To solve optimal control problems [1–4], the averaging method has been used, taking [2–3] the peculiarities of convergence in resonance systems into account [5].

In this paper, unlike the previous approach [1–4], it will be assumed that the purpose of the control is to keep the non-linear system in a small neighbourhood of the resonance ("resonance capture" [5]). Problems of this kind are of interest in the design of control systems operating on the resonance principle [6] or the synchronization principle [7, 8].

It was shown [5] that in the near-resonance region the averaging method is not directly applicable, and the method of successive averaging was introduced to solve the problem in [5]. Various forms of this method have been used [5, 9, 10] to analyse near-resonant systems. Here the successive averaging procedure will be extended for solving optimal control problems in the near-resonance region.

1. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

To simplify the discussion, we will confine our attention to systems in which the slow variable and the controls are scalars [5]. Extension to the multi-dimensional case requires no special proofs.

The equations of motion of a weakly controllable two-frequency system may be written in the form

$$\begin{aligned} x' &= \varepsilon f(x,\theta_1,\theta_2)+\varepsilon^{3/2}F(x,\theta_1,\theta_2,u),\ x(0)=x^0 \\ \theta_1' &= \omega_1(x)+\varepsilon k_1(x,\theta_1,\theta_2)+\varepsilon^{3/2}K_1(x,\theta_1,\theta_2,u),\ \theta_1(0)=\theta_1^0 \\ \theta_2' &= \omega_2(x)+\varepsilon k_2(x,\theta_1,\theta_2)+\varepsilon^{3/2}K_2(x,\theta_1,\theta_2,u),\ \theta_2(0)=\theta_2^0 \end{aligned} \tag{1.1}$$

where $x \in B \subset R_1$, $u \in U$, θ_1, $\theta_2 \in T_2$: $(0, 2\pi) \times (0, 2\pi)$, with U a compact subset in R_1 and $\varepsilon > 0$ a small parameter. The introduction of the parameter $\varepsilon^{3/2}$ in the resonance system will be discussed later. For any admissible control u, the right-hand sides of Eqs (1.1) are assumed to be 2π-periodic in θ_i and sufficiently smooth functions of their variables in the domain $D = \{B \times T_2 \times U\}$; as regards the frequencies, $\omega_{1,2}(x) \geqslant c > 0$ for $x \in B$. It is assumed in addition that the coefficients of Eqs (1.1) contain a finite number of harmonics of the form $\psi_m = r_m\theta_1 + s_m\theta_2$, where r_m and s_m are integers.

We will compare (1.1) with the non-controllable system, which does not contain terms of the order of $\varepsilon^{3/2}$. We will assume that for $x \in B$ the non-controllable system exhibits an isolated primary resonance with the resonance surface

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$$\gamma(x) = r\omega_1(x) + s\omega_2(x) = 0 \tag{1.2}$$

where r and s are certain fixed integers, and a unique isolated solution $\bar{x}$ of Eq. (1.2) exists, that is

$$\gamma(\bar{x}) = 0,\ \gamma_x(\bar{x}) = \Gamma \neq 0$$

We know [5] that the width of the "resonance capture" region B_r is of the order of $\varepsilon^{3/2}$. We will therefore assume that $x^0 = \bar{x} + \mu\rho \in B_r$ at the starting time, and we will consider the motion in a μ-neighbourhood of surface (1.2). Following this approach, we introduce new variables φ and υ characterizing the phase and frequency difference

$$r\theta_1 + s\theta_2 = \varphi,\ \gamma(x) = \mu v \tag{1.3}$$

It follows from (1.3), in particular, that

$$x = x_\mu = x(\mu v) = \bar{x} + \mu\Gamma^{-1}v + \mu^2\ldots \tag{1.4}$$

By virtue of (1.3), we put

$$\theta_1 = \theta,\ \theta_2(\varphi, 0) = s^{-1}(\varphi - r\theta) \tag{1.5}$$

and then represent the coefficients $\{f, k_1, k_2\} = l$ in the form

$$l(x, \theta_1, \theta_2) = l_0(x, \varphi) + l_1(x, \varphi, \theta) \tag{1.6}$$

$$l_0(x,\theta) = \langle l\rangle^\theta = \frac{1}{2\pi s}\int_0^{2\pi s} l(x,\theta,\theta_2(\varphi,\theta))d\theta,\ \langle l_1\rangle^\theta = 0 \tag{1.7}$$

The components of the vector l_1 do not contain resonance harmonics. It is assumed that the averages (1.7) exist uniformly with respect to $x \in B$, $\varphi \in R_1$.

By virtue of (1.4) and (1.5), we can rewrite (1.1) as

$$\begin{aligned}
&v' = \mu\gamma_x(x_\mu)[f(x_\mu,\theta,\theta_2(\varphi,\theta)) + \mu F(x_\mu,\theta,\theta_2(\varphi,\theta),u)]\\
&\varphi' = \mu v + \mu^2 k(x_\mu,\theta,\theta_2(\varphi,\theta)) + \mu^3 K(x_\mu,\theta,\theta_2(\varphi,\theta),u)\\
&\theta' = \omega_1(x_\mu) + \mu^2 k_1(x_\mu,\theta,\theta_2(\varphi,\theta)) + \mu^3 K_1(x_\mu,\theta,\theta_2(\varphi,\theta),u)\\
&k = rk_1 + sk_2,\ K = rK_1 + sK_2
\end{aligned} \tag{1.8}$$

with initial data $\upsilon(0) = \upsilon^0$, $\varphi(0) = \varphi^0$, $\theta(0) = \theta^0$.

Retaining on the right of (1.8) terms of the order of at most two in μ, we obtain

$$\begin{aligned}
&v' = \mu f_0(\varphi,\theta) + \mu^2 f_1(\varphi,\theta)v + \mu^2 F_0(\varphi,\theta,u)\\
&\varphi' = \mu v + \mu^2 k_0(\varphi,\theta),\ \theta' = \omega + \mu\Omega_1 v + \mu^2 Q_2(\varphi,\theta)
\end{aligned} \tag{1.9}$$

where

$$\begin{aligned}
&f_0 = \Gamma f(\bar{x},\theta,\theta_2(\varphi,\theta)),\ F_0 = \Gamma F(\bar{x},\theta,\theta_2(\varphi,\theta),u)\\
&f_1 = \Gamma_1\Gamma^{-1} f(\bar{x},\theta,\theta_2(\varphi,\theta)) + f_x(\bar{x},\theta,\theta_2(\varphi,\theta)), \Gamma_1 = \gamma_{xx}(\bar{x})\\
&k_0 = k(\bar{x},\theta,\theta_2(\varphi,\theta)),\ \omega = \omega_1(\bar{x}),\ \Omega_1 = \Gamma^{-1}\omega
\end{aligned} \tag{1.10}$$

It follows from (1.6) that

$$f_0(\varphi,\theta) = \beta_0(\varphi) + b_1(\varphi,\theta),\ \beta_0(\varphi) = \langle f_0\rangle^\theta,\ \langle b_1\rangle^\theta = 0 \tag{1.11}$$

The function b_1 does not contain resonance harmonics.

For our further analysis, it will be convenient to deal with the slow subsystem

$$v' = \mu\beta_0(\varphi),\ \varphi' = \mu v \tag{1.12}$$

We know [5, 11] that, when investigating typical near-resonance motions, (1.12) may be considered as the equations of motion for an "equivalent pendulum" with potential

$$V(\varphi) = -\int_0^{\varphi} \beta_0(s)ds \tag{1.13}$$

which admits of stable and unstable equilibrium positions. One can distinguish in the phase plane of system (1.12) regions of oscillatory and rotational motion, separated by a separatrix. Then (1.9) may be considered as the equations of the controlled motion of the pendulum driven by fast periodic perturbations.

For the further analysis of Eqs (1.9), we introduce new variables: the total energy of the pendulum

$$e = \upsilon^2/2 + V(\varphi) \tag{1.14}$$

and the rotating phase y, defined by the following relations [12]

$$\begin{gathered} \partial y/\partial\varphi = \Omega(e)/\upsilon(e, \varphi), \ \Omega(e) = 2\pi/T(e) \\ \upsilon(e,\varphi) = \pm[2(e - V(\varphi)]^{1/2}, \ T(e) = \oint \frac{d\varphi}{\upsilon(e,\varphi)} \end{gathered} \tag{1.15}$$

The integration is performed around a contour e = const along a suitable phase trajectory of the unperturbed system. The domain $0 \leqslant e < e_s$ corresponds to oscillatory motion. Here e_s is the energy level corresponding to motion along the separatrix. With this change of variables, $\varphi = \varphi(e, y) = \varphi(e, y + 2\pi)$.

The substitution (1.13)–(1.15) reduces (1.9) to the form

$$\begin{aligned} e' &= \mu B_1(e,\varphi,\theta) + \mu^2\Phi_1(e,\varphi,\theta,u) \\ y' &= \mu\Omega(e) + \mu B_2(e,\varphi,\theta) + \mu^2\Phi_2(e,\varphi,\theta,u) \\ \theta' &= \omega + \mu Q_1(e,\varphi) + \mu^2 Q_2(\varphi,\theta,u) \end{aligned} \tag{1.16}$$

where $\varphi = \varphi(e,y)$, and the right-hand sides of (1.16) are 2π-periodic in y. Here $B_1 = b_1\upsilon$, $B_2 = (\partial y/\partial e)B_1$, that is

$$\langle B_1\rangle^{\theta} = \langle B_2\rangle^{\theta} = 0 \tag{1.17}$$

Consequently, the average rate of change of the function e is of the order of μ^2, not μ, and system (1.16) has a hierarchic structure with slow variable e, a "semi-fast" phase y rotating at a frequency $\mu\Omega(e)$ and a "fast" phase θ rotating at frequency ω. With fixed periodic controls, the approximate solution of (1.16) may be found by successive averaging over θ and y. We will extend this method to solve optimal control problems.

Henceforth we will consider a problem of importance in practice, namely, maintaining a resonance regime in system (1.1). In this formulation, the control problem is to minimize the deviations from the stationary point $\bar{x}$. Let e_μ be the function (1.14), defined on trajectories of the perturbed system (1.8). Considering (1.14) as a measure of deviation from the stationary point, we will write the functional of the problem as

$$J_\mu(u) = M[e_\mu(T)] + \mu^2\int_0^T m(e_\mu, u)dt \tag{1.18}$$

where m and M are fairly smooth functions.

In accordance with the nature of the evolution of the variable e_μ, we consider the dynamics of the system over a time interval of the order of μ^{-2}, that is, the process will end at time $T = \mu^{-2}\sigma$, where $\sigma = O(1)$ [2, 3]. The problem is to construct a control u_μ that will minimize (1.18) on the trajectories of system (1.10). The solution is meaningful if the control system, while in motion, does not leave the region of oscillatory motion, that is, $0 \leqslant e_\mu(t) < e_s$. This inequality is not included in the constraints of the problem, but must be verified when solving the problem.

Equations of the maximum principle. We replace problem (1.8), (1.18) by the simpler problem of minimizing the functional

$$J(u) = M[e(T)] + \mu^2 \int_0^T m(e,u)dt \tag{1.19}$$

on the trajectories of truncated system (1.16). The problem is to construct a control $u_* = \arg\min J(u)/u \in U$ to minimize (1.19) on the trajectories of system (1.16). The relationship between the controls u_μ and u_* will be discussed in Section 5.

We will assume that the optimal control problem (1.16), (1.19) has a solution for $\mu \in (0, \mu_0]$ and that the maximum principle is applicable to it [13]. The Hamiltonian of the problem is

$$H_\mu(e,y,\theta,p,q,\beta,u,\mu) = \mu H + \mu^2 h + \mu q\Omega + \beta[\omega + \mu Q_1 + \mu^2 Q] \tag{1.20}$$

(the arguments on the right of (1.20) are omitted), where

$$H = pB_1 + qB_2, \; h = p\Phi_1 + q\Phi_2 - m \tag{1.21}$$

and it follows from (1.17) that $\langle H \rangle^\theta = 0$.

The required control u_* is determined from the maximum condition for H_μ [13]

$$u_* = \arg\max_{u \in U} H_\mu \tag{1.22}$$

as a function which is sufficiently smooth in all its arguments and is 2π-periodic in y and θ. It follows from (1.20) and (1.22) that

$$u_* = \arg\max_{u \in U} h(e, \varphi(e,y), \theta, p, q, u) = U(e, \varphi(e,y), \theta, p, q) \tag{1.23}$$

Putting $h^* = h(e, \varphi, \theta, p, q, u_*)$, we write the system of equations of the maximum principle for the phase and conjugate elements [13] as

$$\begin{aligned}
&e' = \mu H_p + \mu^2 h_p^* \\
&p' = -\mu H_e - \mu^2 h_e^* - \mu q\Omega_e - \mu\beta Q_{1e} - \mu^2 \beta Q_e \\
&q' = -\mu H_y - \mu^2 h_y^* - \mu\beta Q_{1y} - \mu^2 \beta Q_y, \; \nabla_y = \frac{\partial \varphi}{\partial y}\nabla_\varphi \\
&\beta' = -\mu H_\theta - \mu^2 h_\theta^* - \mu^2 \beta Q_{2\theta} \\
&y' = \mu\Omega + \mu B_2 + \mu^2 \Phi_2, \; \theta' = \omega + \mu Q_1 + \mu^2 Q_2
\end{aligned} \tag{1.24}$$

with boundary conditions

$$\begin{aligned}
&e(0) = e^0, \; y(0) = y^0, \; \theta(0) = \theta^0 \\
&p(T) = -M_e(e(T)), \; q(T) = 0, \; \beta(T) = 0
\end{aligned} \tag{1.25}$$

Here and below, the subscript will denote partial differentiation with respect to the indicated variable.

2. CONSTRUCTION OF THE SOLUTION OF A SYSTEM WITH A HIERARCHY OF PHASE-ROTATION VELOCITIES

To solve system (1.24), (1.25), we will extend the procedure of hierarchic averaging [10] to the analysis of near-resonance motions. Consider the vector $z = (e, p, q, \beta) = \{z^i\}$ $(i = 1, 2, 3, 4)$ and rewrite (1.24) as

$$\begin{aligned}
&z' = \mu Z(z,y,\theta) + \mu^2 S(z,y,\theta), \quad y' = \mu\Omega(e) + \mu Y(e,y,\theta) + \mu^2 C(e,y,\theta) \\
&\theta' = \omega + \mu\Delta(e,y) + \mu^2 \delta(e,y,\theta)
\end{aligned} \tag{2.1}$$

with boundary conditions following from (1.25)

$$F(z(0), z(T)) = 0 \tag{2.2}$$

where Z and S are the vectors of the corresponding components of the first group of equations (1.24); we have used the notation $Y = B_2(e, \varphi(e, y), \theta)$, $\langle Y\rangle^\theta = 0$, $\Delta = Q_1(e, \varphi(e, y))$, etc., in the phase equations. It follows from the relation $\langle H\rangle^\theta = 0$ that $\langle Z\rangle^\theta = 0$. Additionally, it is assumed that the right-hand sides of (2.1) satisfy smoothness conditions that guarantee the validity of the hierarchic averaging procedure [5, 9, 10]. The solution $z(t, \mu)$ will be sought as an expansion [10]

$$z = z_0(\tau) + \mu z_1(\tau, y, \theta) + \mu^2 z_2(\tau, y, \theta) + \mu^3 \dots$$
$$z' = \mu z_{1\theta}\omega + \mu^2(z_{0\tau} + z_{1y}(\Omega + Y) + z_{2\theta}\omega) + \mu^3 \dots, \quad \tau = \mu^2 t \tag{2.3}$$

The coefficients of the expansion must still be 2π-periodic functions of y and θ, which are uniformly bounded with respect to all the variables when $\tau \in [0, \sigma]$, $\sigma = \mu^2 T$. For the coefficients to be uniquely defined, we also require that the condition $\langle z_i\rangle^{\theta y} = 0$ must hold for all higher approximations, $i \geqslant 1$. Then $\langle z\rangle^{\theta y} = z_0 + \mu \dots$. In this problem, we will confine ourselves to determining the coefficients of z_i of at most the second order.

Substituting (2.3) into (2.1) and equating coefficients of like powers of μ, we obtain a system of equations for the successive determination of z_0, z_1 and z_2

$$z_{1\theta}\omega = Z(z_0, y, \theta) \tag{2.4}$$

For the coefficient of z_1 to be periodic in θ, the derivative $z_{1\theta}$ should not contain components that are constant as functions of θ. Hence it follows that

$$z_1(\tau, y, \theta) = \omega^{-1}\int Z(z_0(\tau), y, \theta)d\theta + \zeta_1(z_0(\tau), y) = \tilde{z}_1(z_0, y, \theta) + \zeta_1(z_0, y) \tag{2.5}$$

where the integration is performed with "frozen" values of z_0, y and τ. The function $\zeta_1(z_0, y)$ may be regarded as a constant of integration, which is independent of θ. The conditions $\langle \tilde{z}_1\rangle^\theta = 0$, $\langle \zeta_1\rangle^y = 0$ uniquely define the function $\tilde{z}_1$ and imply the validity of the condition $\langle z_i\rangle^{\theta y} = 0$.

Substituting (2.3) into (2.1) and equating the coefficients of μ^2, we obtain

$$z_{0\tau} = G(z_0, y, \theta) - \omega z_{2\theta}$$
$$G = S + Z_z z_1 - z_{1y}(\Omega + Y) = \tilde{G} + g - \Omega\zeta_{1y} \tag{2.6}$$

where $\langle \tilde{G}\rangle^\theta = 0$. Using the fact that $\langle \zeta_1\rangle^y = 0$, we have

$$g = \langle S\rangle^\theta + \langle Z_z\tilde{z}_1\rangle^\theta - \langle \tilde{Z}_{1y}Y\rangle^\theta = g(z_0, y) \tag{2.7}$$

In turn

$$g(z_0, y) = \tilde{g}(z_0, y) + \gamma(z_0), \quad \gamma = \langle g\rangle^y, \quad \langle \tilde{g}\rangle^y = 0 \tag{2.8}$$

For the functions ζ_1 and z_2 to satisfy the periodicity conditions, we proceed as in (2.4) and set

$$\zeta_{1y} = \Omega^{-1}\tilde{g}(z_0, y), \quad z_{2\theta} = \omega^{-1}\tilde{G}(z_0, y, \theta) \tag{2.9}$$

It then follows from (2.6)–(2.9) that

$$z_{0\tau} = \gamma(z_0) \tag{2.10}$$

Equation (2.10), together with the boundary conditions $F(z_0(0), z_0(\sigma)) = 0$, determine the principal term of the expansion (2.3). Estimates of the accuracy of the solution will be discussed in Section 5.

3. ASYMPTOTIC SOLUTION OF THE SYSTEM OF EQUATIONS OF THE MAXIMUM PRINCIPLE

We will use the procedure just described to solve system (1.24), (1.25). We begin by constructing the expansion for the component $z^4 = \beta = \beta_0 + \mu\beta_1 + \mu_2\beta_2$. From the equation for β, we have

$$Z^4 = -H^0_\theta, \quad S^4 = -h^0_\theta - \beta_0 Q^0_{2\theta} \tag{3.1}$$

The zero superscript indicates that the functions concerned are evaluated at $z = z_0 = (e_0, p_0, q_0, \beta_0)$, Z^i and S^i are the values of the components of the vectors Z and S at $z = z_0$ and the zero superscript of these functions is omitted.

Let us find the function β_0 from Eq. (2.10). Construct the function γ^4 defining the right-hand side of (2.10) and given by relations (2.7) and (2.8). Taking into account that

$$\langle S^4\rangle^\theta = -\langle h_\theta^0\rangle^\theta - \beta_0\langle Q_{2\theta}^0\rangle^\theta \equiv 0$$

we can write (2.7) in the form

$$g(z_0, y) = \langle Z_z^4\tilde{z}_1\rangle^\theta - \langle\tilde{\beta}_{1y}Y^0\rangle^\theta \tag{3.2}$$

By (1.22) and (2.1), we have

$$Z^1 = H_p^0,\ Z^2 = -H_e^0,\ Z^3 = -H_y^0,\ Z^4 = -H_\theta^0,\ Y^0 = B_2^0 = H_q^0 \tag{3.3}$$

It follows from (2.5) and (3.3) that

$$\tilde{e}_1 = K_p^0,\ \tilde{p}_1 = -K_e^0,\ \tilde{q}_1 = -K_y^0,\ \tilde{\beta}_1 = -K_\theta^0 = -H^0 \tag{3.4}$$

where

$$K^0(z_0, y, \theta) = \int H^0(z_0, y, \theta)d\theta,\ \langle K^0\rangle^\theta = 0 \tag{3.5}$$

Substituting (3.3) and (3.4) into (3.2)

$$g^4 = \langle -H_{\theta e}^0K_p^0 + H_{\theta p}^0K_e^0 + H_{\theta q}^0K_y^0 + H_y^0K_q^0\rangle^\theta = \langle -I_1 + I_2 + I_3 + I_4\rangle^\theta \tag{3.6}$$

and evaluating the average by integration by parts, we obtain

$$\langle I_1\rangle^\theta = \langle I_2\rangle^\theta = -\frac{1}{2\pi}\int_0^{2\pi} H_e^0H_p^0d\theta,\ \langle -I_1 + I_2\rangle^\theta = 0$$

(the terms outside the integral vanish by virtue of the periodicity of H^0 and K^0 as functions of θ). From (3.5) and (3.6), we have

$$I_3 + I_4 = \frac{\partial}{\partial\theta}(H_q^0K_y^0)$$

Hence

$$g^4 = \left\langle\frac{\partial}{\partial\theta}(H_q^0K_y^0)\right\rangle^\theta = 0 \tag{3.7}$$

It follows from (2.9), (3.4) and (3.7) that $\beta_1 = \tilde{\beta}_1 = -H^0$ and

$$d\beta_0/d\tau = 0,\ \beta_0(\sigma) = 0 \tag{3.8}$$

giving $\beta_0(\tau) = 0$.

We will construct the expansion $z^3 = q = q_0 + \mu q_1 + \mu^2 q_2$. For $\beta_0 = 0$ we obtain

$$S^3 = -h_y^0 - \beta_1\Delta_y^0,\ g^3 = \langle S^3\rangle^\theta + \langle Z_z^3\tilde{z}_1 - \tilde{q}_{1y}Y^0\rangle^\theta \tag{3.9}$$

For $\langle\beta_1\rangle^\theta = 0$, we have

$$\langle S^3\rangle^\theta = -\langle h_y^0\rangle^\theta,\ \langle S^3\rangle^{\theta y} = -\langle h_y^0\rangle^{\theta y} = 0 \tag{3.10}$$

To calculate the second term in (3.9), we write $g^3 = \langle S^3 + d^3\rangle^\theta$, where, by (3.3), (3.4) and (3.9)

$$d^3 = Z_z^3\tilde{z}_1 - \tilde{q}_{1y}Y^0 = -H_{ye}^0K_p^0 + H_{yp}^0K_e^0 + H_{yq}^0K_y^0 + H_q^0K_{yy}^0 = -I_1 + I_2 + I_3 + I_4 \tag{3.11}$$

Calculating the average $\langle I_1\rangle^\theta$ by integration by parts and taking (3.5) into consideration, we obtain

$$\langle I_1\rangle^\theta=-\frac{1}{2\pi}\int_0^{2\pi}K_{ye}^0H_p^0d\theta$$

$$\langle -I_1+I_2\rangle^\theta=\langle K_{ye}^0H_p^0+H_{yp}^0K_e^0\rangle=\frac{\partial}{\partial y}\langle K_e^0H_p^0\rangle^\theta$$

In turn

$$I_3+I_4=\frac{\partial}{\partial y}(K_y^0H_q^0)$$

It follows from (3.11) and the last few transformations that

$$\langle d^3\rangle^\theta=\frac{\partial}{\partial y}\langle K_e^0H_p^0+K_y^0H_q^0\rangle^\theta,\quad \langle d^3\rangle^{\theta y}=0 \tag{3.12}$$

and, by (3.9)–(3.12)

$$\gamma^3=\langle g^3\rangle^y=\langle d^3\rangle^{\theta y}+\langle S^3\rangle^{\theta y}=0 \tag{3.13}$$

It follows from (2.7) and (3.13) that

$$dq_0/d\tau=\gamma^3=0,\quad q_0(\sigma)=0 \tag{3.14}$$

that is, $q_0(\tau)=0$. At the same time, $q\neq 0$, but $\langle q\rangle^{\theta y}=0$.

Let us construct Eq. (2.10) for e_0 and p_0. It follows from (2.7), (3.3) and (3.4) that

$$\begin{aligned}
&g^i=\langle S^i+d^i\rangle^\theta\\
&S^1=h_p^0,\quad S^2=-h_e^0+\Omega_e^0K_y^0+H^0Q_{1e}^0\\
&d^1=Z_z^1\tilde z_1-Y^0\tilde e_{1y}=H_{pe}^0K_p^0-H_{pp}^0K_e^0-H_{pq}^0K_y^0-H_q^0K_{py}^0\\
&d^2=Z_z^2\tilde z_1-Y^0\tilde p_{1y}=H_{ee}^0K_p^0-H_{ep}^0K_e^0-H_{eq}^0K_y^0-H_q^0K_{ey}^0
\end{aligned} \tag{3.15}$$

Noting that $\langle H^0\rangle^\theta=0$, $\langle K^0\rangle^\theta=0$, we obtain

$$\langle S^1\rangle^{\theta y}=\varkappa_p^0,\quad \langle S^2\rangle^{\theta y}=-\varkappa_e^0,\quad \varkappa^0(e_0,p_0)=\langle h^0\rangle^{\theta y} \tag{3.16}$$

Evaluating $\langle d^{1,2}\rangle^{\theta y}$ using the same transformations as in (3.12), we get

$$\langle d^1\rangle^{\theta y}=\chi_p^0,\quad \langle d^2\rangle^{\theta y}=-\chi_e^0,\quad \chi^0(e_0,p_0)=\langle H_e^0K_p^0+H_y^0K_q^0\rangle^{\theta y} \tag{3.17}$$

Put

$$\eta^0(e_0,p_0)=\varkappa^0(e_0,p_0)+\chi^0(e_0,p_0)=\langle h^0+H_e^0K_p^0+H_y^0K_q^0\rangle^{\theta y} \tag{3.18}$$

Then Eq. (2.10) may be written in the form

$$\begin{aligned}
&de_0/d\tau=\eta_p^0(e_0,p_0),\quad e_0(0)=e^0\\
&dp_0/d\tau=-\eta_e^0(e_0,p_0),\quad p_0(\sigma)=-M_e[e_0(\sigma)]
\end{aligned} \tag{3.19}$$

To determine the precise form of the function η^0, we substitute (1.21) and (3.5) into (3.16)–(3.18) and set $q=0$, $\beta=0$. We thereby obtain

$$\begin{aligned}
&\eta^0(e_0,p_0)=\langle X(e_0,p_0,\varphi(e,y),\theta)\rangle^{\theta y}\\
&X=p[\Phi_1^0+B_{1e}^0R_1^0+B_{1\varphi}^0B_2^0\iota\Omega^{-1}],\quad \partial R_1^0/\partial\theta=B_1^0,\quad \langle R_1^0\rangle^\theta=0
\end{aligned} \tag{3.20}$$

Averaging over φ, we obtain

$$\eta^0(e_0, p_0) = \frac{1}{2\pi T(e_0)} \oint \frac{1}{v(e_0, \varphi)} d\varphi \int_0^{2\pi} X(e_0, p_0, \varphi, \theta) d\theta \tag{3.21}$$

The solution of system (3.19) consistent with the equality $q_0 = 0$ is given by the equation

$$u_0 = \arg\max_{u \in U} h(e_0, \varphi, \theta, p_0, 0, u) = U(e_0, \varphi, \theta, p_0, 0) = U_0(e_0, \varphi, \theta, p_0) \tag{3.22}$$

where h and U are the functions occurring in condition (1.23). It follows from (1.21), (1.23) and (3.22) that the control u_0 is independent of the fast phase θ if the coefficient Φ_1 is independent of θ.

4. ESTIMATE OF THE ACCURACY OF THE SOLUTION

Let u_μ be the control and let $J_\mu(u_\mu)$ be the minimum value of the functional defining the solution of the original problem (1.8), (1.18). Let u_0 be the control (3.20). We will show that

$$0 \leq J_\mu(u_0) - J_\mu(u_\mu) \leq c\mu, \quad \mu \to 0 \tag{4.1}$$

(the left-hand inequality is obvious). Throughout, c will denote constants independent of μ. If condition (4.1) is satisfied, then u_0 is the μ-optimal control in relation to the original problem.

We will make the following assumptions.

1. The right-hand sides of (1.1) and (1.8) are sufficiently smooth and bounded with respect to all variables in the domain D for any admissible control $u \in U$.

2. A solution of each optimal control problem (1.8), (1.18) and (1.16), (1.19) exists and is unique.

3. The right-hand sides of the system of equations of the maximum principle (1.24) satisfies conditions that guarantee the validity of the transformations of the hierarchic averaging method for the Cauchy problem, up to the second approximation [5, 9, 10].

4. The averaged boundary-value problem (3.19) has a unique solution.

It follows from condition 1 [3, 6] that as $\mu \to 0$, $e_\mu(t, \mu)$ asymptotically approaches $e(t)$ over a time interval $t \sim \mu^{-2}$ with accuracy $O(\mu)$ for any admissible control $u \in U$. Hence it follows that the functionals $J_\mu(\mu)$ are close together for any admissible control, in particular

$$|J_\mu(u_*) - J(u_*)| \leq c_1\mu, \quad |J_\mu(u_0) - J(u_0)| \leq c_2\mu, \quad \mu \to 0 \tag{4.2}$$

It follows from (4.2) and from condition 2 [6] that the control u is quasi-optimal with respect to the initial system, that is

$$0 \leq J_\mu(u_*) - J_\mu(u_\mu) \leq c_3\mu, \quad \mu \to 0 \tag{4.3}$$

It follows from conditions 3 and 4 [3, 6] not only that the solutions e and e_0 are close together, but also that the control u_0 is quasi-optimal with respect to the truncated system (1.18)

$$0 \leq J(u_0) - J(u_*) \leq c_4\mu, \quad \mu \to 0 \tag{4.4}$$

Using (4.2)–(4.4), we write

$$\begin{aligned} 0 \leq J_\mu(u_0) - J_\mu(u_\mu) &= [J_\mu(u_0) - J(u_0)] + [J(u_0) - J(u_*)] + \\ &+ [J(u_*) - J_\mu(u_*)] + [J_\mu(u_*) - J_\mu(u_\mu)] \leq (c_1 + c_2 + c_3 + c_4)\mu \leq c\mu \end{aligned} \tag{4.5}$$

which is identical with (4.1).

Remarks. 1. We have here considered only control with a fixed instant of completion of the process. Using analogous reasoning, we can construct approximate solutions of time-optimal problems. Suppose that the purpose of the control is to drive the system from its initial state $e_\mu(0) = e^0$ to an equilibrium position $e_\mu(T) = 0$ in the minimum time $T = \mu^{-2}\sigma$, in such a way as to minimize functional (1.18) ($G = 0$), subject to constraints on the control $u \in U$. In that case a quasi-optimal control u_0 is defined by (3.20), where $e_0(\tau)$ and $p_0(\tau)$ constitute a solution of system (3.19) satisfying the boundary conditions $e_0(0) = e^0$, $e_0(\sigma) = 0$. An additional condition that must be satisfied is $\eta^0 = 1$, $\tau = \sigma$ (compare [3]).

2. It follows from Sections 2 and 3 that the problem of controlling near-resonance oscillations may be regarded as a special case of a more general problem, namely, control in systems with different phase rotation velocities.

5. EXAMPLE

Let us consider the problem of vibrationally maintaining the rotation of an unbalanced rotor [7]. The simplest model may be represented by a mathematical pendulum of mass m and length l whose pivot is moving vertically as given by the relation $s(t) = a\cos\omega t$. It is well known [7] that such a system admits of a stable state of uniform rotation at frequency ω. Suppose the system is also subject to a rotational torque $M(t)$, $|M| \leqslant M_0$, whose purpose is to drive the system from a rotational state with initial frequency ω^0 to a synchronous state in the least possible time.

Assuming that the amplitudes of the vibration of the base are small compared with the dimensions of the pendulum, we introduce a small parameter $\varepsilon = \mu^2 = a/2l$. We also assume that $\omega \gg k$, where $k = (g/l)^{1/2}$ is an eigenfrequency of weak free oscillations of the pendulum and $(k/\omega)^2 = \mu^3\gamma$. Assuming that the system is weakly controllable, we set

$$\mu^3 u = M/ml^2\omega^2, \quad |u| \leqslant m_0, \quad \mu^3 m_0 = M_0/ml^2\omega^2$$

Then the equation of motion may be written in the form

$$\theta_1'' + \mu^2 \sin\theta_1(\mu\gamma + 2\cos\theta_2) = \mu^3 u \tag{5.1}$$

where θ_1 is the angle between the pendulum and its lower equilibrium position, $\theta_2 = \omega t$ is the driving phase, and the prime denotes differentiation with respect to the dimensionless variable ωt.

Equation (5.1) may be reduced to standard form (1.1)

$$\begin{aligned} &x' = -\mu^2(\mu\gamma\sin\theta_1 + 2\sin\theta_1\cos\theta_2) + \mu^2 u, \quad x(0) = x^0 = \omega^0/\omega \\ &\theta_1' = x, \quad \theta_2' = 1, \quad \theta_1(0) = 0, \quad \theta_2(0) = 0 \end{aligned} \tag{5.2}$$

Considering the motion in a small neighbourhood of the frequency resonance $x = 1$, we construct the change of variables (1.7)

$$\theta_1 - \theta_2 = \varphi, \quad x - 1 = \mu\upsilon, \quad \theta_2 = \theta \tag{5.3}$$

and rewrite (5.2) in the form of (1.1)

$$\begin{aligned} &v' = -\mu\sin\varphi - \mu[\sin(\varphi + 2\theta) + \mu\gamma\sin(\varphi + \theta)] + \mu^2 u, \quad v(0) = v^0 \\ &\varphi' = \mu v, \quad \theta' = 1 \\ &\beta_0(\varphi) = -\sin\varphi, \quad V(\varphi) = 1 - \cos\varphi, \quad v^0 = \mu^{-1}(\omega/\omega_0 - 1) \end{aligned} \tag{5.4}$$

Introducing new variables e and y as in (1.14) and (1.15), we obtain

$$\begin{aligned} &e' = -\mu[\sin(\varphi + 2\theta) + \mu\gamma\sin(\varphi + \theta)]v + \mu^2 uv \\ &y' = \mu\Omega(e) + \mu B(e, y, \theta) + \mu^2\Phi_2(e, y, \theta, u) \\ &\theta' = 1 \\ &v(e, \varphi) = \pm[2(e - 1 + \cos\varphi)]^{1/2} \\ &B_2 = y_e\sin(\varphi + 2\theta), \quad \Phi_2 = -\gamma y_e\sin(\varphi + \theta) \end{aligned} \tag{5.5}$$

The control u_0 is defined by condition (3.20). By (1.21), (3.20) and (5.5), we have

$$h^0 = p_0 uv(e_0, \varphi), \quad u_0 = \arg\max_{|u| \leqslant m_0} up_0 v = m_0 \operatorname{sign}(p_0 v) \tag{5.6}$$

Employing arguments similar to those in [2, 3], we obtain $\operatorname{sign} p_0 = -1$, that is, the control synthesis has the form

$$u_0 = -m_0 \operatorname{sign} v, \quad M = -M_0 \operatorname{sign}(\theta' - 1) \tag{5.7}$$

corresponding to well-known solutions of problems involving the damping of pendulum-type systems [2, 3].

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